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A Special Class of 2-Varieties

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INTRODUCTION

Let $\mathscr{V}_F$ be a variety of power-associative algebras over a field F that satisfy the condition that there exists an integer $s \geqslant 2$ such that $A \in \mathscr{V}_F$ and I ideal of A implies I^s ideal of A. Such varieties have been called s-varieties by Zwier [9]. Many authors have studied s-varieties for $s = 2$. Examples of 2-varieties are associative, alternative, and Lie algebras. Anderson in [2] has shown that a variety $\mathscr{V}_F$ is a 2-variety if and only if all algebras of $\mathscr{V}_F$ satisfy the following two identities

$$\begin{aligned}(x_1x_2)x_3 &= \alpha_1(x_3x_1)x_2 + \alpha_2(x_1x_3)x_2 + \alpha_3x_2(x_3x_1) + \alpha_4x_2(x_1x_3)\\ &\quad + \alpha_5(x_3x_2)x_1 + \alpha_6(x_2x_3)x_1 + \alpha_7x_1(x_3x_2) + \alpha_8x_1(x_2x_3),\\ x_3(x_1x_2) &= \beta_1(x_3x_1)x_2 + \beta_2(x_1x_3)x_2 + \beta_3x_2(x_3x_1) + \beta_4x_2(x_1x_3)\\ &\quad + \beta_5(x_3x_2)x_1 + \beta_6(x_2x_3)x_1 + \beta_7x_1(x_3x_2) + \beta_8x_1(x_2x_3),\end{aligned} \tag{1}$$

where the α's and the β's are assumed to be in F.

In 1949 Albert in [1] gave a classification, up to quasi-equivalence, of those 2-varieties that satisfy the further condition of the existence in $\mathscr{V}_F$ of a noncommutative algebra with identity. Anderson and Kleinfeld in [3] have given a classification of 2-varieties considering not only the case studied by Albert but also those 2-varieties which contain a nonzero finite-dimensional semisimple nil algebra and in [4] they have studied in detail a special class of 2-varieties. In a forthcoming paper [7] Hentzel and the present author give a complete classification of (not necessarily power-associative) 2-varieties by studying the defining identities by matrix techniques. In the present paper this same technique is used to study in detail the

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class of (not necessarily power-associative) 2-varieties that contain a noncommutative algebra with identity and it is shown that every algebra R in this class is either associator dependent or the following identity

$$(a+b-1)[[x_1, x_3], x_2] + (2a-b)\{(x_2, x_1, x_3) - (x_2, x_3, x_1)\}$$
$$+ (x_1, x_3, x_2) - (x_3, x_1, x_2) = 0, \qquad 2a - b \neq \pm 1$$

together with the cyclic law $(x_1, x_2, x_3) + (x_2, x_3, x_1) + (x_3, x_1, x_2) = 0$ holds in R or in its anti-isomorphic copy where multiplication is defined as $a \cdot b = ba$. We thus improve Anderson and Kleinfeld's result in [3] reducing their classification to only one type. Also, we point out which are the possible types of associator dependent algebras belonging to a 2-variety that contains a noncommutative algebra with identity.

It is worthwhile emphasizing the strength of this technique when dealing with classification problems of identities. Through this technique one can draw consequences of the defining identities that would otherwise be difficult to obtain.

Notations and Definitions

Throughout the paper by R we will mean a nonassociative algebra over a field F of characteristic not 2 or 3.

The associator (a, b, c) and commutator $[a, b]$ are defined as usual by $(a, b, c) = (ab)c - a(bc)$ and $[a, b] = ab - ba$.

For the matrix representation technique of the identities we refer to [5] or to [6]. Since we will be dealing with identities of degree three, we will make use of the group ring G over F of the symmetric group S_3 and the fact that $G \simeq F \oplus F \oplus F_{2\times 2}$, where $F_{2\times 2}$ represents the 2×2 matrices over F. When $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} c & d \\ 0 & 0 \end{smallmatrix}\right)$ is an identity of R in the 2×2 representation based on $RR \cdot R \oplus R \cdot RR$, we will say that (a, b, c, d) is an identity of R in the 2×2 representation.

In this paper by saying that an algebra is associator dependent we mean that it satisfies a nonzero identity of the form $(a, b, -a, -b)$ in the 2×2 representation. Thus, Lie admissibility $(a, b, c) + (b, c, a) + (c, a, b) - (a, c, b) - (c, b, a) - (b, a, c) = 0$ in this paper is not considered an associator dependent identity. Also, for associator dependency we do not require third power associativity.

Representation of 2-Varieties $\mathscr{V}_F$

Since a variety of algebras is a 2-variety if and only if it satisfies the two identities (1), it is easily seen that its representation, based on the functions $f = RR \cdot R$ and $g = R \cdot RR$, is the following:

	$RR \cdot R$	$\oplus$	$R \cdot RR$
$\mathscr{R}_1$	$-2w_1 + w_2 - 3$ $-2z_1 + z_2$		$w_5 + w_6$ $z_5 + z_6 - 3$
$\mathscr{R}_2$	$-w_2 - 2w_4 - 3$ $-z_2 - 2z_4$		$w_6 - w_5 - 2w_7 + 2w_8$ $z_6 - z_5 - 2z_7 + 2z_8 - 3$
$\mathscr{R}_3$	$\begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$		$\begin{pmatrix} w_5 & w_6 \\ w_7 & w_8 \end{pmatrix}$ $\begin{pmatrix} z_5 & z_6 \\ z_7 & z_8 \end{pmatrix}$

where

$$\begin{aligned}
w_1 &= -1 - \alpha_1 - \alpha_5 & z_1 &= -\beta_1 - \beta_5 \\
w_2 &= -\alpha_1 + \alpha_2 - \alpha_5 + \alpha_6 & z_2 &= -\beta_1 + \beta_2 - \beta_5 + \beta_6 \\
w_3 &= \alpha_1 + \alpha_2 - \alpha_6 & z_3 &= \beta_1 + \beta_2 - \beta_6 \\
w_4 &= -1 + \alpha_5 - \alpha_6 & z_4 &= \beta_5 - \beta_6 \\
w_5 &= \alpha_4 + \alpha_8 & z_5 &= 1 + \beta_4 + \beta_8 \\
w_6 &= \alpha_3 + \alpha_7 & z_6 &= 1 + \beta_3 + \beta_7 \\
w_7 &= -\alpha_3 - \alpha_4 + \alpha_7 & z_7 &= -1 - \beta_3 - \beta_4 + \beta_7 \\
w_8 &= -\alpha_3 - \alpha_4 + \alpha_8 & z_8 &= -\beta_3 - \beta_4 + \beta_8.
\end{aligned}$$

We will from now on restrict our attention to 2-varieties $\mathscr{V}_F$ that contain a noncommutative algebra with identity: we will indicate such varieties by $\bar{\mathscr{V}}_F$. Setting $x_1 = x_2 = x_3 = 1$ in (1) implies

$$\sum_{i=1}^{8} \alpha_i = \sum_{i=1}^{8} \beta_i = 1.$$

If we then let $x_i = 1$ $(i = 1, 2, 3)$, by using the previous relations one has

$$(\alpha_1 + \alpha_2 + \alpha_7 + \alpha_8 - 1)[x_1, x_2] = 0,$$

$$(\alpha_2 + \alpha_4 + \alpha_7 + \alpha_8 - 1)[x_1, x_3] = 0,$$

$$(\alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 - 1)[x_2, x_3] = 0,$$

and analogous relations for the β's. The existence of a noncommutative algebra in the variety and the relations between the α's and the w's and between the β's and the z's imply that the following relations must hold:

$$\begin{aligned} -2w_1+w_2+w_5+w_6&=3,\\ -2w_1+w_2-2w_3+w_4-w_7-w_8&=0,\\ 2w_1-w_2-w_3-w_4-2w_7+w_8&=0,\\ 9+7w_1-2w_2+w_3+w_4-3w_6+2w_7-w_8&=0. \end{aligned}$$

The same relations hold for the z's. Solving both systems yields

$$\begin{aligned} w_5&=-w_1, & w_6&=3+3w_1-w_2,\\ z_5&=-z_1, & z_6&=3+3z_1-z_2,\\ w_7&=-w_3, & w_8&=-2w_1+w_2-w_3+w_4,\\ z_7&=-z_3, & z_8&=-2z_1+z_2-z_3+z_4. \end{aligned}$$

The representation of $\bar{\mathscr{V}}_F$ is therefore the following:

	$RR\cdot R$	$\oplus$ $R\cdot RR$
$\mathscr{R}_1$	$-2w_1+w_2-3$ $-2z_1+z_2$	$2w_1-w_2+3$ $2z_1-z_2$
$\mathscr{R}_2$	$-w_2-2w_4-3$ $-z_2-2z_4$	w_2+2w_4+3 z_2+2z_4
$\mathscr{R}_3$	$\begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$	$\begin{pmatrix} -w_1 & 3+3w_1-w_2 \\ -w_3 & -2w_1+w_2-w_3+w_4 \end{pmatrix}$ $\begin{pmatrix} -z_1 & 3+3z_1-z_2 \\ -z_3 & -2z_1+z_2-z_3+z_4 \end{pmatrix}.$

Classification

We will now classify varieties $\bar{\mathscr{V}}_F$ depending on the rank of the 2×2 representation $\mathscr{R}_3$, starting with rank 1, since it is immediately seen that rank 0 cannot occur.

(a) Representation $\mathscr{R}_3$ has rank 1. This means that λ, μ, ν, and $\tau\in F$ must exist such that

$$\begin{array}{lll} w_1 = \lambda a & w_2 = \lambda b & 3 + 3w_1 - w_2 = \lambda d \\ w_3 = \mu a & w_4 = \mu b & -2w_1 + w_2 - w_3 + w_4 = \mu d \\ z_1 = \nu a & z_2 = \nu b & 3 + 3z_1 - z_2 = \nu d \\ z_3 = \tau a & z_4 = \tau b & -2z_1 + z_2 - z_3 + z_4 = \tau d \end{array}$$

For this to happen we must have $3a - b - d \neq 0$, $a - b + d \neq 0$,

$$\lambda = \nu = \frac{-3}{3a - b - d}, \qquad \mu = \tau = \frac{3(2a - b)}{(3a - b - d)(a - b + d)}.$$

The corresponding representation will then be

	$RR \cdot R$	$\oplus$	$R \cdot RR$	
$\mathscr{R}_1$	$\frac{3(-a+d)}{3a-b-d}$		$\frac{3(a-d)}{3a-b-d}$	
	$\frac{3(2a-b)}{3a-b-d}$		$\frac{3(-2a+b)}{3a-b-d}$	
$\mathscr{R}_2$	$\frac{-3(a+d)}{a-b+d}$		$\frac{3(a+d)}{a-b+d}$	
	$\frac{-3b}{a-b+d}$		$\frac{3b}{a-b+d}$	
$\mathscr{R}_3$	$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} -a & d \\ 0 & 0 \end{pmatrix}$	$3a \neq b + d$ $a \neq b - d.$

By $3a \neq b + d$ and $a \neq b - d$ we thus have

	$RR \cdot R$	$\oplus$	$R \cdot RR$	
$\mathscr{R}_1$	1		-1	
$\mathscr{R}_2$	1		-1	
$\mathscr{R}_3$	$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} -a & d \\ 0 & 0 \end{pmatrix}$	$3a \neq b + d$ $a \neq b - d.$

(b) Representation $\mathscr{R}_3$ has rank $\geqslant 2$. We will show (I. R. Hentzel) that this case implies a nonidentically zero associator dependent identity. Let

$$\begin{array}{cccc} a & b & -a & d, \\ a' & b' & -a' & d', \end{array}$$

be two independent rows of $\mathscr{R}_3$.

If $\left|\begin{smallmatrix} b & d \\ b' & d' \end{smallmatrix}\right| \neq 0$, then $(b'+d')(a, b, -a, d)-(b+d)(a', b', -a', d')$, i.e., $(ab'+ad'-ba'-da', \; d'b-db', \; -b'a-d'a+ba'+da', b'd-bd')$ is an associator dependent identity which by $bd'-b'd \neq 0$ is not identically zero.

If $\left|\begin{smallmatrix} b & d \\ b' & d' \end{smallmatrix}\right| = 0$, then the two identities are

$$\begin{array}{cccc} a & b & -a & d, \\ a' & \lambda b & -a' & \lambda d, \end{array}$$

whence $(a'-\lambda a, 0, -a'+\lambda a, 0)$ is nonidentically zero associator dependent identity, since $a'-\lambda a \neq 0$, or else the two identities are not independent.

We have proved the following theorem:

THEOREM. *Let $\mathscr{V}_F$ be a 2-variety which contains a noncommutative algebra with identity. Then, if the algebras in the variety are not all associator dependent, then its representation is*

	$RR \cdot R$	$\oplus$	$R \cdot RR$	
$\mathscr{R}_1$	1		-1	
$\mathscr{R}_2$	1		-1	
$\mathscr{R}_3$	$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} -a & d \\ 0 & 0 \end{pmatrix}$	$3a \neq b+d$, $a \neq b-d$.

COROLLARY. *Every 2-variety which contains a noncommutative algebra with identity and is not associator dependent satisfies the cyclic law $(x_1, x_2, x_3)+(x_2, x_3, x_1)+(x_3, x_1, x_2)=0$. Also, every 2-variety (associator dependent or not) containing a noncommutative algebra with identity and having representation $\mathscr{R}_3$ of rank 1 satisfies the cyclic law.*

Proof. It is a consequence of the representation of the cyclic law (see Table III, p. 57 of [6]). We should always keep in mind that by associator dependency we mean in this paper an identity of the form $(a, b, -a, -b)$ in $\mathscr{R}_3$, based on the functions $RR \cdot R$ and $R \cdot RR$.

We now ask the following question: which are the possible types of associator dependent algebras that form a 2-variety containing a non commutative algebra with identity? Let us consider the class $\mathscr{A}$ of all associator dependent algebras in our sense, i.e., the algebras which have representation, based on the function $f=(R, R, R)$,

	(R, R, R)
$\mathscr{R}_1$	0
$\mathscr{R}_2$	0
$\mathscr{R}_3$	$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$.

Within this class we are seeking the algebras that form a 2-variety containing a noncommutative algebra with identity. Since all algebras in $\mathscr{A}$ have rank 1 in $\mathscr{R}_3$ for them to be a 2-variety by the Corollary to the Theorem they need to satisfy the cyclic law. Also, by $3a \neq b + d$ and $a \neq b - d$ of (a), their representation will be

	(R, R, R)	
$\mathscr{R}_1$	1	
$\mathscr{R}_2$	1	
$\mathscr{R}_3$	$\begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix}$	$\lambda \neq \frac{1}{2}$.

COROLLARY. *Flexible and antiflexible algebras do not form a 2-variety* $\bar{\mathscr{V}}_F$.

Proof. The excluded cases of the previous representation are

	(R, R, R)			(R, R, R)
$\mathscr{R}_1$	1		$\mathscr{R}_1$	1
$\mathscr{R}_2$	1	and	$\mathscr{R}_2$	1
$\mathscr{R}_3$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		$\mathscr{R}_3$	$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$

which correspond, respectively, to flexibility $(a, b, a) = 0$ (*together with Lie-admissibility*) *and antiflexibility* $(a, b, c) - (c, b, a) = 0$ (*together with third-power associativity*).

IDENTITIES AND COMPARISON WITH OTHER WORK

We have seen that if the algebras in a 2-variety containing a non commutative algebra with identity are not all associator dependent, then their representation is the following:

	$RR \cdot R$	$\oplus$	$R \cdot RR$	
$\mathscr{R}_1$	1		-1	
$\mathscr{R}_2$	1		-1	
$\mathscr{R}_3$	$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} -a & d \\ 0 & 0 \end{pmatrix}$	$3a \neq b+d$ $a \neq b-d$

This corresponds to the following identity:

$$(b+d)[[x_1, x_3], x_2] + (a-d)\{(x_1, x_2, x_3) + (x_1, x_3, x_2)\} \\ + (-a+b-d)(x_2, x_3, x_1) + (2a-b)(x_2, x_1, x_3) = 0$$

together with the cyclic law $(x_1, x_2, x_3) + (x_2, x_3, x_1) + (x_3, x_1, x_2) = 0$ and $d \neq 3a-b$, $d \neq b-a$.

Depending on whether $a=d$ or $a \neq d$ this identity becomes:

(i) $(a+b)[[x_1, x_3], x_2] + (2a-b)\{(x_2, x_1, x_3) - (x_2, x_3, x_1)\} = 0$, $2a-b \neq 0$ or

(ii) $(a+b-1)[[x_1, x_3], x_2] + (2a-b)\{(x_2, x_1, x_3) - (x_2, x_3, x_1)\}$
$+ (x_1, x_3, x_2) - (x_3, x_1, x_2) = 0 \quad 2a-b \neq \pm 1$.

Now, if we redefine multiplication in R as $a \cdot b = ba$, then identity (i) becomes

(i′) $\alpha[[x_1, x_3], x_2] + (x_1, x_3, x_2) - (x_3, x_1, x_2) = 0$.

(i′) together with the cyclic law is easily seen to be (ii) for $b = 2a$.

We thus have the following theorem:

THEOREM. *Let $\mathscr{V}_F$ be a 2-variety containing a noncommutative algebra with identity. Then either every algebra of $\mathscr{V}_F$ is associator dependent, or it is isomorphic or anti-isomorphic to an algebra satisfying* (ii) *together with the cyclic law.*

Identity (i) corresponds to

$$\alpha[[x_1, x_3], x_2] = (x_2, x_3, x_1) - (x_2, x_1, x_3) \qquad \text{for} \quad \alpha = \frac{a+b}{2a-b}$$

which is (1.6) of [3].

Identity (ii) is (1.10) of [3] for $\delta = b - 2a$ and $\alpha_1 = (a+b-1)/(2a-b+1)(-2a+b+1)$. Identity (1.7) of [3] implies flexibility, so that in presence of the cyclic law it turns out to be (1.6). In presence of the cyclic law identity (1.8) of [3] is (1.10) for $\delta = 0$, and identity (1.9) is (1.10) for $\alpha_1 = (2\delta + 1)/2(1 - \delta^2)$. We finally observe that in [4] the five possible types had been reduced with respect to [3] in some cases, also through anti-isomorphism considerations. Our result reduces them to only one type.

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